

The General PBW Property*

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Abstract. For ungraded quotients of an arbitrary \mathbb{Z} -graded ring, we define the general PBW property, that covers the classical PBW property and the N -type PBW property studied via the N -Koszulity by several authors ([BG1], BG2], [FV]). In view of the noncommutative Gröbner basis theory, we conclude that every ungraded quotient of a path algebra (or a free algebra) has the general PBW property. We remark that an earlier result of Golod [Gol] concerning Gröbner bases can be used to give a homological characterization of the general PBW property in terms of Shafarevich complex. Examples of application are given.

Key words PBW Property, graded algebra, Gröbner basis ¹

0. Introduction

Let $K\langle X \rangle$ be the free associative algebra on a set of noncommuting variables X over a field K , and let $K\langle X \rangle = \bigoplus_{p \in \mathbb{N}} K\langle X \rangle_p$ be the decomposition of $K\langle X \rangle$ by its homogeneous components $K\langle X \rangle_p$ spanned by words of length $p \geq 0$. Then $K\langle X \rangle$ has the natural filtration $FK\langle X \rangle = \{F_p K\langle X \rangle\}_{p \in \mathbb{N}}$ with $F_p K\langle X \rangle = \bigoplus_{i \leq p} K\langle X \rangle_i$. For a K -subspace $P \subset F_N K\langle X \rangle$, $N \geq 2$, let $\langle P \rangle$ be the two-sided ideal of $K\langle X \rangle$ generated by P and write $A = K\langle X \rangle / \langle P \rangle$. Then $FK\langle X \rangle$ induces a filtration $FA = \{F_p A\}_{p \in \mathbb{N}}$ on A , where $F_p A = (F_p K\langle X \rangle + \langle P \rangle) / \langle P \rangle$, that defines the associated graded K -algebra $G(A) = \bigoplus_{p \in \mathbb{N}} G(A)_p$ with $G(A)_p = F_p A / F_{p-1} A$. Since

$$\frac{K\langle X \rangle_p \oplus F_{p-1} K\langle X \rangle}{(F_p K\langle X \rangle \cap \langle P \rangle) + F_{p-1} K\langle X \rangle} = \frac{F_p K\langle X \rangle}{(F_p K\langle X \rangle \cap \langle P \rangle) + F_{p-1} K\langle X \rangle} \xrightarrow{\cong} G(A)_p, \quad p \in \mathbb{N},$$

there is the natural graded epimorphism $\phi: K\langle X \rangle \rightarrow G(A)$. On the other hand, consider $P_N = \{f \in P \mid f \notin F_{N-1} K\langle X \rangle\} \subseteq P$. Then every $f \in P_N$ has a unique presentation $f =$

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$f_N + f_{N-1} + \dots + f_{N-s} \in P$ with $f_N \in K\langle X \rangle_N$, $f_{N-j} \in K\langle X \rangle_{N-j}$ and $f_N \neq 0$. Write $\mathbf{LH}(f) = f_N$ for each $f \in P_N$. Then $\mathbf{LH}(f) \in (\langle P \rangle \cap F_N K\langle X \rangle) + F_{N-1} K\langle X \rangle$. Thus, if $\langle \mathbf{LH}(P_N) \rangle$ denotes the graded two-sided ideal of $K\langle X \rangle$ generated by $\mathbf{LH}(P_N) = \{ \mathbf{LH}(f) \mid f \in P_N \}$, then $\langle \mathbf{LH}(P_N) \rangle \subseteq \text{Ker} \phi$. It follows that the canonical graded epimorphism $\pi: K\langle X \rangle \rightarrow \overline{A} = K\langle X \rangle / \langle \mathbf{LH}(P_N) \rangle$ yields naturally a graded epimorphism $\rho: \overline{A} \rightarrow G(A)$ such that the following diagram commutes

$$\begin{array}{ccc} K\langle X \rangle & \xrightarrow{\pi} & \overline{A} \\ \phi \downarrow & \swarrow \rho & \\ G(A) & & \end{array}$$

Actually, the property that ρ is an isomorphism is an analogue of the classical PBW (abbreviation of Poincarè-Birkhoff-Witt) theorem for enveloping algebras of Lie algebras. For $N = 2$, Braverman and Gaitsgory [BG1] studied this isomorphism problem posed by Joseph Bernstein. Applying graded deformations to both graded hochschild cohomology and Koszul algebras, they obtained a PBW theorem as follows.

Theorem A ([BG1] Theorem 0.5) Suppose that P satisfies

- (I) $P \cap F_1 K\langle X \rangle = \{0\}$ and
- (J) $(F_1 K\langle X \rangle \cdot P \cdot F_1 K\langle X \rangle) \cap F_2 K\langle X \rangle = P$.

If the quadratic algebra $\overline{A} = K\langle X \rangle / \langle \mathbf{LH}(P_2) \rangle$ is Koszul in the classical sense, then ρ is an isomorphism.

If we call the PBW property studied in [BG1] the *2-type PBW property* for the reason that $P \subset F_2 K\langle X \rangle$, then generally for $N \geq 2$, the *N-type PBW property* was studied in the very recent work [FV] and [BG2] respectively. Gunnar Floystad and Jon Eivind Vatne dealt with the *N-type PBW property* in [FV] for deformations of *N-Koszul K-algebras* and the obtained *N-type PBW theorem* states that

Theorem B ([FV] Theorem 4.1) Suppose that the graded algebra $\overline{A} = K\langle X \rangle / \langle \mathbf{LH}(P_N) \rangle$ is an *N-Koszul algebra* in the sense of [Ber]. Then ρ is an isomorphism if and only if

$$\langle P \rangle \cap F_N K\langle X \rangle = P.$$

While Roland Berger and Victor Ginzburg dealt with the *N-type PBW property* in [BG2] for ungraded quotients of the tensor algebra over a Von Neumann regular ring K and an *N-type PBW theorem* was obtained as well.

Theorem C ([BG2] Theorem 3.4) Suppose that P satisfies

- (a) $P \cap F_{N-1} K\langle X \rangle = \{0\}$ and
- (b) $(P \cdot K\langle X \rangle_1 + K\langle X \rangle_1 \cdot P) \cap F_N K\langle X \rangle = P$.

If the graded left K -module $\mathrm{Tor}_3^A(K_A, {}_A K)$ is concentrated in degree $N + 1$, then ρ is an isomorphism.

As a consequence, an extension of the N -Koszulity [Ber] to nonhomogeneous algebras was realized through the N -type PBW property in [BG2].

Remark Note that we have stated Theorems A–C in the language of the present paper. For instance, in [FV], the algebra \overline{A} is a given N -Koszul algebra defined by homogeneous elements of degree N , and the algebra A is a deformation of \overline{A} such that its associated graded algebra is exactly \overline{A} .

In this paper, for ungraded quotients of an arbitrary \mathbb{Z} -graded ring, we define first the general PBW property, that covers the classical PBW property and the N -type PBW property studied via the N -Koszulity in the literature. This is reached in section 1 after a clear picture of ungraded quotients vs graded quotients is established (Theorem 1.6). In section 2, we focus on ungraded quotients of path algebras (including free algebras) and realize the general PBW property by means of Gröbner bases. We remark in section 3 that an earlier result of Golod [Gol] concerning Gröbner bases can be used to give a homological characterization of the general PBW property (for positively graded algebra) in terms of Shafarevich complex. Finally in section 4, some examples of applications of sections 1–2 are discussed.

Here we point out that the main idea and principal method used in section 1 and section 2 were announced in Chapter III of [Li], where similar results were discussed only for quotients of finitely generated free algebras but the general PBW property for ungraded quotients of graded algebras was not exposed.

Throughout this paper, by a graded ring we mean an associative \mathbb{Z} -graded ring with unity. Let $B = \bigoplus_{p \in \mathbb{Z}} B_p$ be a graded ring. If $B_i = 0$ for all $i < 0$, then we say that B is positively graded and write $B = \bigoplus_{p \in \mathbb{N}} B_p$. We adopt the conventional notion on graded rings and call an element $F_p \in B_p$ a homogeneous element of degree p . Thus, if $f = f_p + f_{p-1} + \cdots + f_{p-s}$ with $f_p \in B_p$, $F_{p-j} \in B_{p-j}$ and $f_p \neq 0$, then we say that f has degree p and write $d(f) = p$. Let I be an ideal of B . Then I is a graded ideal if and only if $I = \bigoplus_{p \in \mathbb{Z}} (B_p \cap I)$ if and only if the quotient ring $B/I = \bigoplus_{p \in \mathbb{Z}} (B_p + I/I)$. Unless otherwise stated, all graded ring (module) homomorphisms are of degree 0.

1. The General PBW Property

In this section we introduce and characterize the general PBW property for ungraded quotients of an arbitrary \mathbb{Z} -graded ring. Since such a property is defined for filtered rings, from both a structural and a computational viewpoint (see Proposition 1.7 and Theorem 2.2), it is natural to bring both the associated graded ring and the Rees ring into the data considered. To begin

with, let us review some necessary results on filtered rings and their associated graded objects.

Let A be a \mathbb{Z} -filtered associative ring with filtration FA :

$$\cdots \subset F_{p-1}A \subset F_pA \subset F_{p+1}A \subset \cdots, \quad p \in \mathbb{Z},$$

where each F_pA is an abelian subgroup of A (if A is a K -algebra over some commutative ring K , then F_pA is a K -submodule of A) such that $1 \in F_0A$, $A = \cup_{p \in \mathbb{Z}} F_pA$, and $F_pAF_qA \subset F_{p+q}A$ for all $p, q \in \mathbb{Z}$. FA induces two graded structures, that is, the associated graded ring $G(A)$ of A which is defined as $G(A) = \oplus_{p \in \mathbb{Z}} G(A)_p$ with $G(A)_p = F_pA/F_{p-1}A$, and the Rees ring \tilde{A} of A which is defined as $\tilde{A} = \oplus_{p \in \mathbb{Z}} F_pA$. Write X for the homogeneous element of degree 1 in $\tilde{A}_1 = F_1A$ represented by 1, which is usually called the *canonical element* of \tilde{A} . Then X is contained in the center of \tilde{A} and is not a divisor of 0. Consider the ideal $\langle 1 - X \rangle = (1 - X)\tilde{A}$, respectively $\langle X \rangle = X\tilde{A}$, of \tilde{A} generated by $1 - X$, respectively by X . Then it is well-known that

$$(*) \quad \tilde{A}/\langle 1 - X \rangle \cong A, \quad \tilde{A}/X\tilde{A} \cong G(A).$$

On the other hand, Let $B = \oplus_{n \in \mathbb{Z}} B_n$ be a graded ring and X a homogeneous element of degree 1, that is, $X \in B_1$. Suppose that X is contained in the center of B and is not a divisor of 0. Put $\Lambda = B/\langle 1 - X \rangle$, where $\langle 1 - X \rangle$ is the ideal of B generated by $1 - X$. Note that if $b_p \in B_p$ is a homogeneous element of degree p , then $Xb_p \in B_{p+1}$ because X is of degree 1. Thus $b_p = Xb_p + (1 - X)b_p$ implies $B_p + \langle 1 - X \rangle \subset B_{p+1} + \langle 1 - X \rangle$. Consequently, the \mathbb{Z} -gradation on B induces naturally a \mathbb{Z} -filtration $F\Lambda$ on Λ :

$$F_n\Lambda = \frac{B_p + \langle 1 - X \rangle}{\langle 1 - X \rangle}, \quad p \in \mathbb{Z}.$$

1.1. Proposition (see [LVO]) With notation as above, the following statements hold.

- (i) $\langle 1 - X \rangle$ does not contain any nonzero homogeneous element of B .
- (ii) The associated graded ring $G(\Lambda)$ of Λ with respect to $F\Lambda$ is isomorphic to B/XB under graded ring homomorphism.
- (iii) The Rees ring $\tilde{\Lambda}$ of Λ with respect to $F\Lambda$ is isomorphic to B under graded ring homomorphism. In particular, X corresponds to the canonical element of $\tilde{\Lambda}$.

□

For the remainder of this section, let $R = \oplus_{n \in \mathbb{Z}} R_n$ be an arbitrary graded ring.

Consider the polynomial ring $R[t]$ over R in one commuting variable t . Then the onto ring homomorphism $\phi: R[t] \rightarrow R$ defined by $\phi(t) = 1$ has $\text{Ker}\phi = \langle 1 - t \rangle$, the ideal of $R[t]$ generated by $1 - t$. Hence $R \cong R[t]/\langle 1 - t \rangle$. Since $R[t]$ has the mixed gradation, that is, $R[t] = \oplus_{p \in \mathbb{Z}} R[t]_p$ with

$$R[t]_p = \left\{ \sum_{i+j=p} F_i t^j \mid F_i \in R_i, j \geq 0 \right\}, \quad p \in \mathbb{Z},$$

for each $f \in R$, there exists a homogeneous element $F \in R[t]_p$, for some p , such that $\phi(F) = f$. More precisely, if $f = f_p + f_{p-1} + \cdots + f_{p-s}$ where $f_p \in R_p$, $f_{p-j} \in R_{p-j}$ and $f_p \neq 0$, then $f^* = f_p + t f_{p-1} + \cdots + t^s f_{p-s}$ is a homogeneous element in $R[t]_p$ satisfying $\phi(f^*) = f$.

1.2. Definition (i) For any $F \in R[t]$, write $F_* = \phi(F)$. F_* is called the *dehomogenization* of F with respect to t .

(ii) For an element $f \in R$, if $f = f_p + f_{p-1} + \cdots + f_{p-s}$ with $f_p \in R_p$, $f_{p-j} \in R_{p-j}$ and $f_p \neq 0$, then the homogeneous element $f^* = f_p + t f_{p-1} + \cdots + t^s f_{p-s}$ in $R[t]_p$ is called the *homogenization* of f with respect to t .

(iii) If I is a two-sided ideal of R , then we let $\langle I^* \rangle$ stand for the graded two-sided ideal of $R[t]$ generated by $I^* = \{f^* \mid f \in I\}$. $\langle I^* \rangle$ is called the *homogenization ideal* of I with respect to t .

With definition and notation as above, the following 1.3–1.4 may be found in [Li].

1.3. Lemma (i) For $F, G \in R[t]$, $(F + G)_* = F_* + G_*$, $(FG)_* = F_* G_*$.

(ii) For $f, g \in R$, $(fg)^* = f^* g^*$, $t^s(f + g)^* = t^r f^* + t^h g^*$, where $r = d(g)$, $h = d(f)$, and $s = r + h - d(f + g)$.

(iii) For any $f \in R$, $(f^*)_* = f$.

(iv) If F is a homogeneous element of degree p in $R[t]$, and if $(F_*)^*$ is of degree q , then $t^r(F_*)^* = F$, where $r = p - q$.

(v) If I is a two-sided ideal of R , then each homogeneous element $F \in \langle I^* \rangle$ is of the form $t^r f^*$ for some $r \in \mathbb{N}$ and $f \in I$.

□

1.4. Proposition Let I be a proper two-sided ideal of R . Then the map

$$\begin{aligned} \alpha : R[t]/\langle I^* \rangle &\longrightarrow R/I \\ F + \langle I^* \rangle &\mapsto F_* + I, \end{aligned} \quad F \in R[t]$$

is an onto ring homomorphism with $\text{Ker } \alpha = \langle 1 - \bar{t} \rangle$, where \bar{t} denotes the coset of t in $R[t]/\langle I^* \rangle$. Moreover, \bar{t} is not a divisor of 0 in $R[t]/\langle I^* \rangle$, and hence $\langle 1 - \bar{t} \rangle$ does not contain any nonzero homogeneous element of $R[t]/\langle I^* \rangle$.

□

Consider the natural grading filtration FR on R which is defined by the the abelian subgroups

$$F_p R = \oplus_{i \leq p} R_i, \quad p \in \mathbb{Z}.$$

Let I be a proper two-sided ideal of R and $A = R/I$. Then FR induces the quotient filtration FA on A :

$$F_p A = (F_p R + I)/I, \quad p \in \mathbb{Z},$$

that defines two graded structures: the associated graded ring $G(A) = \bigoplus_{p \in \mathbb{Z}} G(A)_p$ of A with $G(A)_p = F_p A / F_{p-1} A$, and the Rees ring $\tilde{A} = \bigoplus_{p \in \mathbb{Z}} \tilde{A}_p$ of A with $\tilde{A}_p = F_p A$. The proposition below shows that $G(A)$ and \tilde{A} may be determined by $\langle I^* \rangle$.

1.5. Proposition With notation as before, there are graded ring isomorphisms:

- (i) $\tilde{A} \cong R[t] / \langle I^* \rangle$, and
- (ii) $G(A) \cong R[t] / (\langle t \rangle + \langle I^* \rangle)$, where $\langle t \rangle$ denotes the ideal of $R[t]$ generated by t .

Proof Put $B = R[t] / \langle I^* \rangle$, $B_p = (R[t]_p + \langle I^* \rangle) / \langle I^* \rangle = \overline{R[t]_p}$, $p \in \mathbb{Z}$. Then \bar{t} is a homogeneous element of degree 1 in B , and by Proposition 1.4, it is not a divisor of 0. Hence B is isomorphic to the Rees ring $\tilde{\Lambda}$ of the filtered ring $\Lambda = B / \langle 1 - \bar{t} \rangle$, where

$$F_p \Lambda = \frac{B_p + \langle 1 - \bar{t} \rangle}{\langle 1 - \bar{t} \rangle} = \frac{\overline{R[t]_p} + \langle 1 - \bar{t} \rangle}{\langle 1 - \bar{t} \rangle}, \quad p \in \mathbb{Z},$$

and moreover, $B / \bar{t} B \cong G(\Lambda)$. On the other hand, it is not difficult to see that the ring homomorphism $\alpha: R[t] / \langle I^* \rangle \rightarrow R / I = A$ defined in Proposition 1.4 yields isomorphisms of abelian groups:

$$F_p \Lambda = \frac{\overline{R[t]_p} + \langle 1 - \bar{t} \rangle}{\langle 1 - \bar{t} \rangle} \longrightarrow \frac{\bigoplus_{i \leq p} R_i + I}{I} = F_p A, \quad p \in \mathbb{Z},$$

which extend to define a graded ring isomorphism

$$\tilde{\alpha}: B \cong \tilde{\Lambda} = \bigoplus_{p \in \mathbb{Z}} \frac{\overline{R[t]_p} + \langle 1 - \bar{t} \rangle}{\langle 1 - \bar{t} \rangle} \longrightarrow \bigoplus_{p \in \mathbb{Z}} F_p A = \tilde{A}.$$

But note that under α we have $t + \langle I^* \rangle \mapsto 1 + I$. Thus, under the graded ring isomorphism $\tilde{\alpha}$ we have $\bar{t} \mapsto X$, the canonical element of \tilde{A} . It follows from the formula (*) given in the beginning of this section and Proposition 1.1 that (i) and (ii) hold. \square

Further, we present $G(A)$ as a graded quotient of R by finding its defining ideal clearly. To this end, for $f \in R$ we denote by $\mathbf{LH}(f)$ the *leading homogeneous part* of f , that is, if $f = f_p + f_{p-1} + \cdots + f_{p-s}$ with $f_p \in R_p$, $f_{p-j} \in R_{p-j}$ and $f_p \neq 0$, then $\mathbf{LH}(f) = f_p$. Thus, if S is a subset of R , then we put

$$\mathbf{LH}(S) = \left\{ \mathbf{LH}(f) \mid f \in S \right\}$$

and write $\langle \mathbf{LH}(S) \rangle$ for the graded two-sided ideal generated by $\mathbf{LH}(S)$ in R .

Since $G(A) = \bigoplus_{p \in \mathbb{Z}} G(A)_p$, where $G(A)_p = F_p A / F_{p-1} A$ with $F_p A = (F_p R + I) / I$, there are canonical isomorphisms of abelian groups

$$(1) \quad \frac{R_p \oplus F_{p-1} R}{(I \cap F_p R) + F_{p-1} R} = \frac{F_p R}{(I \cap F_p R) + F_{p-1} R} \xrightarrow{\cong} G(A)_p, \quad p \in \mathbb{Z}.$$

It follows that the natural epimorphisms of abelian groups

$$\phi_p: R_p \longrightarrow \frac{R_p \oplus F_{p-1} R}{(I \cap F_p R) + F_{p-1} R}, \quad p \in \mathbb{Z},$$

extend to define a graded epimorphism

$$\phi: R \longrightarrow G(A).$$

1.6. Theorem With the convention made above, we have $\text{Ker}\phi = \langle \mathbf{LH}(I) \rangle$, and hence $G(A) \cong R/\langle \mathbf{LH}(I) \rangle$.

Proof It is sufficient to prove the equalities

$$\text{Ker}\phi_p = \langle \mathbf{LH}(I) \rangle \cap R_p, \quad p \in \mathbb{Z}.$$

Suppose $f_p \in \text{Ker}\phi_p$. Then $f_p \in (I \cap F_p R) + F_{p-1}R$. If $f_p \neq 0$, then as f_p is a homogeneous element of degree p , we have $f_p = \mathbf{LH}(f)$ for some $f \in I \cap F_p R$. This shows that $f_p \in \langle \mathbf{LH}(I) \rangle \cap R_p$. Hence $\text{Ker}\phi_p \subseteq \langle \mathbf{LH}(I) \rangle \cap R_p$. Conversely, suppose $f_p \in \langle \mathbf{LH}(I) \rangle \cap R_p$. Then $f_p = \sum g_i \mathbf{LH}(f_i) h_i$, where g_i and h_i are homogeneous elements. Let $f_i = \mathbf{LH}(f_i) + f'_i$ where $\deg(f'_i) < \deg(f_i)$. Then $f_p = \sum g_i f_i h_i - \sum g_i f'_i h_i$ with $\sum g_i f_i h_i \in I \cap F_p R$ and $\sum g_i f'_i h_i \in F_{p-1}R$. This shows that $f_p \in (I \cap F_p R) + F_{p-1}R$, that is, $f_p \in \text{Ker}\phi_p$. Hence, $(\langle \mathbf{LH}(I) \rangle \cap R_p) \subseteq \text{Ker}\phi_p$. Summing up, we conclude the desired equalities. \square

Now, let \mathcal{F} be an arbitrary subset of the ideal I . Then by the foregoing discussion,

$$\langle \mathbf{LH}(\mathcal{F}) \rangle \subseteq \langle \mathbf{LH}(I) \rangle = \text{Ker}\phi.$$

It follows that the canonical graded epimorphism $\pi: R \rightarrow \bar{A} = R/\langle \mathbf{LH}(\mathcal{F}) \rangle$ yields naturally a graded epimorphism $\rho: \bar{A} \rightarrow G(A)$ such that the following diagram commutes

$$\begin{array}{ccc} R & \xrightarrow{\pi} & \bar{A} \\ \phi \downarrow & \swarrow \rho & \\ G(A) & & \end{array}$$

If we set $R = K\langle X \rangle$, $I = \langle P \rangle$, and $\mathcal{F} = P_N$ as in section 0, then the property that ρ is an isomorphism gives exactly the *N-type PBW Property*. Instead of giving our definition of the general PBW property immediately by using the phrase “ ρ is an isomorphism”, let us see first how Theorem 1.6 reveals the essential feature of this property.

1.7. Proposition Let \mathcal{F} be an arbitrary subset of the ideal I and $\bar{A} = R/\langle \mathbf{LH}(\mathcal{F}) \rangle$. With the convention made above, the following statements are equivalent.

- (i) The natural graded epimorphism $\rho: \bar{A} \rightarrow G(A)$ is an isomorphism.
- (ii) $\langle \mathbf{LH}(I) \rangle = \langle \mathbf{LH}(\mathcal{F}) \rangle$.
- (iii) \mathcal{F} is a set of generators for I that has the property: every $f \in I$ has a presentation $f = \sum g_j f_j h_j$, where $g_j, h_j \in R$ and $f_j \in \mathcal{F}$, such that $d(g_j) + d(f_j) + d(h_j) \leq \deg(f)$ for all $g_j f_j h_j \neq 0$.

(iv) $\langle I^* \rangle = \langle \mathcal{F}^* \rangle$ and hence $\tilde{A} \cong R[t]/\langle \mathcal{F}^* \rangle$, where $\mathcal{F}^* = \{f^* \mid f \in \mathcal{F}\}$.

Proof (i) \Leftrightarrow (ii) By the construction of ρ , this equivalence is clear.

(ii) \Leftrightarrow (iii) Suppose $\langle \mathbf{LH}(\mathcal{F}) \rangle = \langle \mathbf{LH}(I) \rangle$. If $f \in I$ with $d(f) = p$, then since $\mathbf{LH}(f)$ is a homogeneous element we have $\mathbf{LH}(f) = \sum g_j \mathbf{LH}(f_j) h_j$ for some homogeneous elements $g_j, h_j \in R$, $f_j \in \mathcal{F}$, and $d(g_j) + d(\mathbf{LH}(f_j)) + d(h_j) = d(g_j) + d(f_j) + d(h_j) = p$. Now the element $f' = f - \sum g_j f_j h_j \in I$ has $d(f') < p$, so we may repeat the same procedure for f' . Since $d(f) = p$ is finite, after a finite number of reduction steps we obtain a presentation $f = \sum g_j f_j h_j$ where g_j, h_j are homogeneous elements of R , $f_j \in \mathcal{F}$ and $d(g_j) + d(f_j) + d(h_j) \leq p$ for all i . It follows that (iii) holds.

Conversely, suppose (iii) holds. Then it is easy to see that for any $f \in I$, $\mathbf{LH}(f) = \sum \mathbf{LH}(g_j) \mathbf{LH}(f_j) \mathbf{LH}(h_j)$ for some $g_j, h_j \in R$, $f_j \in \mathcal{F}$. Hence $\langle \mathbf{LH}(\mathcal{F}) \rangle = \langle \mathbf{LH}(I) \rangle$.

(iv) \Leftrightarrow (iii) To prove this equivalence, first recall and bear in mind that if $f_p + f_{p-1} + \cdots + f_{p-s} = f \in R$ with $f_p \in R_p$, $f_{p-j} \in R_{p-j}$ and $f_p \neq 0$, then $f^* = f_p + t f_{p-1} + \cdots + t^s f_{p-s}$. Consequently, $d(f) = d(f^*) = p$ and $\mathbf{LH}(f) = \mathbf{LH}(f^*) = f_p$.

Suppose (iv) holds. Then for $f \in I$ with $d(f) = p$, we have $f^* \in \langle I^* \rangle$. Hence, $f^* = \sum G_j f_j^* H_j$, in which $f_j^* \in \mathcal{F}^*$, G_j and H_j are homogeneous elements of $R[t]$ and $d(G_j) + d(f_j^*) + d(H_j) = p$ whenever $G_j f_j^* H_j \neq 0$. It follows from Lemma 1.3 that

$$f = (f^*)_* = \sum G_{j*} (f_j^*)_* H_{j*} = \sum G_{j*} f_j H_{j*},$$

where $d(G_{j*}) + d(f_j) + d(H_{j*}) \leq p$ whenever $G_{j*} f_j H_{j*} \neq 0$. This shows that (iii) holds.

Conversely, suppose (iii) holds. To reach (iv), we need only to consider homogeneous elements. If $F \in \langle I^* \rangle$ is a homogeneous element, then by Lemma 1.3, $F = t^r f^*$ for some integer $r \geq 0$ and some $f \in I$. Suppose $f = \sum_j h_j f_j g_j$. Then $d(h_j^*) + d(f_j^*) + d(g_j^*) \leq d(f^*)$. We may use Lemma 1.3 and the assumption (iii) to start a reduction procedure as follows.

Begin

$$f^* - \sum_j h_j^* f_j^* g_j^* = t^{r_1} m_1^* + t^{r_2} m_2^* + \cdots \text{ with } r_j > 0, m_j \in I, \text{ and } d(t^{r_j} m_j^*) \leq d(f^*) \text{ for all } m_j.$$

For each $m_j^* \in I^*$, where $m_j = \sum_i h_{ij} f_{ij} g_{ij}$, since $d(h_{ij}^*) + d(f_{ij}^*) + d(g_{ij}^*) \leq d(m_j^*)$, so go to

Next

$$m_j^* - \sum_i h_{ij}^* f_{ij}^* g_{ij}^* = t^{r_{1j}} m_{1j}^* + t^{r_{2j}} m_{2j}^* + \cdots \text{ with } r_{kj} > 0, m_{kj} \in I, \text{ and } d(t^{r_{kj}} m_{kj}^*) \leq d(m_j^*) \text{ for all } m_{kj}.$$

As $d(f^*)$ is finite, after a finite number of steps we may reach $f^* \in \langle \mathcal{F}^* \rangle$, in particular, $f^* = \sum_j h_j^* f_j^* g_j^*$ with $d(h_j^*) + d(f_j^*) + d(g_j^*) \leq d(f^*)$ for all j . (Since the ideal I considered should be a proper ideal, the dehomogenization operation on $R[t]$ guarantees that the final result of the reduction procedure cannot be an expression like $\sum_\ell t^\ell$.) This proves the conclusion of (iv). \square

Proposition 1.7 tells us that if ρ is an isomorphism, then the subset \mathcal{F} is necessarily a set of generators for the ideal I , that is, \mathcal{F} is not really “arbitrary”.

1.8. Definition Let R , I and $A = R/I$ be as before, and let \mathcal{F} be a set of generators for the ideal I . The ring A is said to have the *general PBW property*, if one of the equivalent conditions in Proposition 1.7 is satisfied.

Remark (i) Clearly, Definition 1.8 covers the N -type PBW property, and it is also obvious that if I is a graded ideal, then this definition becomes trivial. If I is not a graded ideal, then, just like verifying the sufficient conditions for the N -type PBW property in Theorems A–C of [BG1], [FV] and [BG2], any of the equivalent conditions in Proposition 1.7 is not easy to be verified. We will see in next section that for ideals of a path algebra (or a free algebra), Gröbner bases with respect to a certain gradation-preserving monomial ordering can realize Proposition 1.7 effectively.

(ii) Suppose that I is ungraded, or equivalently, $A = R/I$ is not a graded ring. Then, except reaching a unified definition for the PBW property, the importance of Theorem 1.6 may also be indicated from a viewpoint of lifting structures. For instance, if R is a finitely generated free algebra or a finitely generated path algebra, and if the ring $R/\langle \mathbf{LH}(I) \rangle$ is one of the following type: a domain, a Noetherian ring, an Artinian ring, a graded semisimple ring, a ring with finite global dimension, an Auslander regular ring, a ring with classical standard PBW-basis, etc, then $A = R/I$ is a ring of the same type at the ungraded level, and moreover, all properties listed may be lifted to the Rees ring \tilde{A} of A (see [LVO]).

2. Gröbner Basis Means The General PBW Property

In this section, we realize the general PBW property for quotients of path algebras (including free algebras) by means of Gröbner bases. In principle, as the *Noncommutative Buchberger Algorithm* ([Mor], [Gr]) produces a (finite or infinite) Gröbner basis for each two-sided ideal of a path algebra (or a free algebra), we may say, from both a theoretical and a practical viewpoint, that every ungraded quotient of a path algebra (or a free algebra) has the general PBW property.

Before starting the main text of this section, let us explain briefly why path algebra is our first choice. Let K be a field and Q a finite directed graph (or a quiver). Recall that the path algebra KQ is defined to be the K -algebra with the K -basis the set of finite directed paths in Q , where the vertices of Q are viewed as paths of length 0, and the multiplication in KQ is induced by multiplication of paths. Note that the free associative K -algebra on n noncommuting variables is isomorphic to the path algebra KQ where Q has one vertex and n loops, and hence, every finitely generated K -algebra is of the form KQ/I , where I is a two-sided ideal of KQ . It is well-known from representation theory that every finite dimensional K -algebra is Morita equivalent to an algebra of the form KQ/I if K is algebraically closed; and since KQ has the natural gradation defined by the lengths of the paths, quotients of path algebra over K include graded K -algebras $A = \bigoplus_{i \geq 0} A_i$, where A_0 is a product of a finite number of copies of K , each A_i is a finite dimensional K -vector space and A is generated in degree 0 and 1, that is, for $i, j \geq 0$,

$A_i A_j = A_{i+j}$. It is also known that an algebra is N -Koszul in the sense of [Ber] if and only if it is a quotient of a path algebra by an ideal generated by homogeneous elements of degree N and its Yoneda algebra is generated in degree 0, 1 and 2. Thus, our choice of path algebra has a big generality. In particular, every path algebra KQ holds a well-developed Gröbner basis theory. So, defining relations of a quotient of KQ may be studied algorithmically, and this advantage enables us to reach the main result of this section.

For a general theory on noncommutative Gröbner bases, the reader is referred to, for example, [Mor], [Gr] and [Li].

To maintain the notation of section 1, let us write $R = KQ$ and use the natural positively graded structure $R = \bigoplus_{p \in \mathbb{N}} R_p$ on R , where the gradation is defined by the lengths of paths in R . Let I be a two-sided ideal of R and $A = R/I$. Then A has the filtration FA induced by the grading filtration FR on R . Let $G(A)$ and \tilde{A} be the associated graded algebra and the Rees algebra of A defined by FA , respectively. Then by Proposition 1.5 and Theorem 1.6, there are isomorphisms of graded K -algebras: $G(A) \cong R/\langle \mathbf{LH}(I) \rangle$, $\tilde{A} \cong R[t]/\langle I^* \rangle$.

From now on in this section we fix an admissible system $(R, \mathcal{B}, \succeq_{gr})$, that is, \mathcal{B} is the K -basis of R consisting of monomials (finite directed paths), and \succeq_{gr} is some *graded monomial ordering* on \mathcal{B} , for example, the graded lexicographic ordering. If $f \in R$, $f = \sum \lambda_i u_i$ where $\lambda_i \in K$ and $u_i \in \mathcal{B}$, then we write $\mathbf{LM}(f) = \max\{u_i \mid \lambda_i \neq 0\}$ for the *leading monomial* of f . For a subset S of R we put

$$\mathbf{LM}(S) = \left\{ \mathbf{LM}(f) \mid f \in S \right\}$$

and write $\langle \mathbf{LM}(S) \rangle$ for the two-sided monomial ideal of R generated by $\mathbf{LM}(S)$. Recall that a subset $\mathcal{G} \subset I$ is called a *Gröbner basis* for the two-sided ideal I if

$$\langle \mathbf{LM}(I) \rangle = \langle \mathbf{LM}(\mathcal{G}) \rangle.$$

2.1. Theorem Let \mathcal{G} be a Gröbner basis for the two-sided ideal I in R with respect to \succeq_{gr} , and $A = R/I$. With notation maintained from section 1, the following statements hold.

- (i) $\langle \mathbf{LH}(I) \rangle = \langle \mathbf{LH}(\mathcal{G}) \rangle$, and hence $G(A) \cong R/\langle \mathbf{LH}(\mathcal{G}) \rangle$, that is, the algebra A has the general PBW property in the sense of Definition 1.8.
- (ii) $\langle I^* \rangle = \langle \mathcal{G}^* \rangle$, and hence $\tilde{A} \cong R[t]/\langle \mathcal{G}^* \rangle$.

Proof Since \mathcal{G} is a Gröbner basis for I , it is well-known that for $f \in I$, starting with $\mathbf{LM}(f)$, the equality $\langle \mathbf{LM}(I) \rangle = \langle \mathbf{LM}(\mathcal{G}) \rangle$ (or a division on f by \mathcal{G}) yields inductively a Gröbner presentation

$$f = \sum \lambda_j u_j g_j v_j, \quad \lambda_j \in K, \quad u_j, v_j \in \mathcal{B}, \quad g_j \in \mathcal{G},$$

in which $u_j \mathbf{LM}(g_j) v_j \neq 0$ and $\mathbf{LM}(f) \succeq_{gr} \mathbf{LM}(u_j g_j v_j)$. But note that the monomial ordering \succeq_{gr} preserves gradation. It follows that $\mathbf{LM}(f)$ comes from $\mathbf{LH}(f)$ and they have the same degree at the graded level. Consequently, in the Gröbner presentation of f obtained above we

also have

$$d(f) \geq d(u_j) + d(g_j) + d(v_j) \text{ for all } u_j g_j v_j.$$

This shows that Proposition 1.7(iii) is satisfied. Hence (i) and (ii) hold. \square

In computational algebra it is a well-known fact that, starting with a set of homogeneous elements, a homogeneous Gröbner basis may be obtained in a more effective way. At this point, in addition to its own independent interest, the next theorem will be helpful in realizing the general PBW property algorithmically.

Consider the k -basis

$$\mathcal{B}(t) = \left\{ wt^r \mid w \in \mathcal{B}, r \geq 0 \right\}$$

for $R[t]$. Then the monomial ordering \succeq_{gr} on \mathcal{B} extends to a monomial ordering on $\mathcal{B}(t)$, again denoted \succeq_{gr} , as follows:

$$w_1 t^{r_1} \succ_{gr} w_2 t^{r_2} \text{ if and only if } w_1 \succ_{gr} w_2, \text{ or } w_1 = w_2 \text{ and } r_1 > r_2.$$

With the definition made above, we have $X_j \succ_{gr} t^r$ for all $j \in \Lambda$ and all $r \geq 0$, and a Gröbner basis theory holds in $R[t]$ exactly as in R .

2.2. Theorem Let I be a two-sided ideal of R and $\mathcal{G} \subset I$. With notation as before, the following statements are equivalent:

- (i) \mathcal{G} is a Gröbner basis of I in R ;
- (ii) $\mathbf{LH}(\mathcal{G})$ is a Gröbner basis of $\langle \mathbf{LH}(I) \rangle$ in R .
- (iii) \mathcal{G}^* is a Gröbner basis of $\langle I^* \rangle$ in $R[t]$;

Proof First note that \succeq_{gr} and the definition of homogenization yield the following equalities:

$$(**) \quad \begin{cases} \mathbf{LM}(f) = \mathbf{LM}(\mathbf{LH}(f)), & f \in R, \\ \mathbf{LM}(f^*) = \mathbf{LM}(f), & f \in R. \end{cases}$$

(i) \Leftrightarrow (ii) Every element of $\langle \mathbf{LH}(I) \rangle$ has a presentation of the form $\sum \lambda_i u_i \mathbf{LH}(f_i) v_i$, where $\lambda_i \in K$, $u_i, v_i \in \mathcal{B}$ and $f_i \in I$. Consequently, by the formula (**) above, the desired equivalence follows from the equivalence below: for $f \in I$, $g_j \in \mathcal{G}$,

$$\mathbf{LM}(f) = u \mathbf{LM}(g_j) v \Leftrightarrow \mathbf{LM}(\mathbf{LH}(f)) = u \mathbf{LM}(\mathbf{LH}(g_j)) v.$$

(i) \Leftrightarrow (iii) Suppose (i) holds. Noticing that $\langle I^* \rangle$ is a graded ideal, it needs only to consider homogeneous elements of $\langle I^* \rangle$. Let $F \in \langle I^* \rangle$ be a nonzero homogeneous element. Then by Lemma 1.3, $F = t^r f^*$, where $r \geq 0$ and $f \in R$. It follows from the foregoing formula (**) that

$$\begin{aligned} \mathbf{LM}(F) &= t^r \mathbf{LM}(f^*) \\ &= t^r \mathbf{LM}(f) \\ &= t^r u \mathbf{LM}(g_j) v, \text{ for some } u, v \in \mathcal{B}, g_j \in \mathcal{G}, \\ &= t^r u \mathbf{LM}(g_j^*) v. \end{aligned}$$

This shows that $\langle \mathbf{LM}(\langle I^* \rangle) \rangle \subseteq \langle \mathbf{LM}(\mathcal{G}^*) \rangle$, and hence the equality holds. Therefore, \mathcal{G}^* is a Gröbner basis of $\langle I^* \rangle$.

Conversely, suppose (iii) holds. Then for any $f \in I$, by the formula (*) we have

$$\begin{aligned} \mathbf{LM}(f) &= \mathbf{LM}(f^*) \\ &= u\mathbf{LM}(g_j^*)v, \text{ for some } u, v \in \mathcal{B}, g_j \in \mathcal{G}, \\ &= u\mathbf{LM}(g_j)v. \end{aligned}$$

This shows that $\langle \mathbf{LM}(I) \rangle \subseteq \langle \mathbf{LM}(\mathcal{G}) \rangle$, and hence the equality holds. Therefore, \mathcal{G} is a Gröbner basis of I . \square

3. A Characterization in Terms of Shafarevich Homology

In this section, we remark that the general PBW property for *positively* \mathbb{N} -graded algebra can be characterized by the first homology of the Shafarevich complex. This is based on an earlier work of Golod [Gol] in which *standard bases* (including Gröbner bases in path algebras and free algebras) was studied by means of Shafarevich homology and the classical Koszulity was involved in the commutative case. To understand this, note first that for a positively \mathbb{N} -graded K -algebra $R = \bigoplus_{p \in \mathbb{N}} R_p$, where K is a field, if we adopt the notion and notation of [Gol] by setting $\Gamma = \mathbb{N}$ and using the grading \mathbb{N} -filtration FR as the Γ -filtration, then the property $\langle \mathbf{LH}(I) \rangle = \langle \mathbf{LH}(\mathcal{F}) \rangle$ (Proposition 1.7(iii)) is just an analogue of the definition for a *standard basis* (including Gröbner basis) \mathcal{F} in R . It turns out that our general PBW property also has a homological characterization, as to which, we mention now as follows.

Let $X = \{x_j\}_{j \in J}$, and let U denote the free K -algebra $K\langle X \rangle$ on the set X . By definition, the *Shafarevich complex* relative to F , denoted $\text{Sh}(X|\mathcal{F}, R)$, is a complex of R - R -bimodules

$$\text{Sh}_n(X|\mathcal{F}, R) = \underbrace{R \otimes U \otimes R \otimes \cdots \otimes R \otimes U}_{n \text{ copies of } R \otimes U} \otimes R, \quad n \geq 0,$$

(tensor product is defined over K) and differentials

$$d_n : \text{Sh}_n(X|\mathcal{F}, R) \longrightarrow \text{Sh}_{n-1}(X|\mathcal{F}, R)$$

with

$$\begin{aligned} d_n(a_0 \otimes x_{j_1} \otimes a_1 \otimes \cdots \otimes x_{j_{i-1}} \otimes a_{i-1} \otimes x_{j_i} \otimes a_i \otimes x_{j_{i+1}} \otimes \cdots \otimes a_{n-1} \otimes x_{j_n} \otimes a_n) \\ = \sum_{i=1}^n (-1)^{i-1} a_0 \otimes x_{j_1} \otimes a_1 \otimes \cdots \otimes x_{j_{i-1}} \otimes (a_{i-1} f_{j_i} a_i) \otimes x_{j_{i+1}} \otimes \cdots \otimes a_{n-1} \otimes x_{j_n} \otimes a_n, \end{aligned}$$

Consider the grading filtration FR on R as before. Then FR induces a \mathbb{N} -filtration $F\text{Sh}(X|F, R)$ on $\text{Sh}(X|\mathcal{F}, R)$, where for each p , $n \geq 0$, $F_p \text{Sh}_n(X|\mathcal{F}, R)$ is the K -subspace spanned by

$$a_0 \otimes x_{j_1} \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes x_{j_n} \otimes a_n$$

in which a_i s are homogeneous elements such that

$$\deg(a_0) + \deg(f_{j_1}) + \deg(a_1) + \cdots + \deg(a_{n-1}) + \deg(f_{j_n}) + \deg(a_n) \leq p.$$

With respect to this filtered structure, d_n is a filtered homomorphism of degree 0, and hence, $\text{Sh}(X|\mathcal{F}, R)$ becomes a \mathbb{N} -filtered complex. It follows that there are two associated \mathbb{N} -graded complexes $G(\text{SH})(X|\mathcal{F}, R)$ and $\widetilde{\text{Sh}}(X|\mathcal{F}, R)$, where

$$\bigoplus_{p \in \mathbb{N}} \frac{F_p \text{Sh}_n(X|\mathcal{F}, R)}{F_{p-1} \text{Sh}_n(X|\mathcal{F}, R)} = G(\text{Sh})_n(X|\mathcal{F}, R) \xrightarrow{G(d_n)} G(\text{Sh})_{n-1}(X|\mathcal{F}, R) = \bigoplus_{p \in \mathbb{N}} \frac{F_p \text{Sh}_{n-1}(X|\mathcal{F}, R)}{F_{p-1} \text{Sh}_{n-1}(X|\mathcal{F}, R)},$$

and

$$\bigoplus_{p \in \mathbb{N}} F_p \text{Sh}_n(X|\mathcal{F}, R) = \widetilde{\text{Sh}}_n(X|\mathcal{F}, R) \xrightarrow{\tilde{d}_n} \widetilde{\text{Sh}}_{n-1}(X|\mathcal{F}, R) = \bigoplus_{p \in \mathbb{N}} F_p \text{Sh}_{n-1}(X|\mathcal{F}, R).$$

Put $\mathbf{LH}(\mathcal{F}) = \{\mathbf{LH}(f_j)\}_{j \in J}$. Then there is the Shafarevich complex $\text{Sh}(X|\mathbf{LH}(\mathcal{F}), R)$ relative to $\mathbf{LH}(\mathcal{F})$ with differentials D_n , $n \geq 0$. Now, the natural graded surjective morphism

$$\varphi: \text{Sh}(X|\mathbf{LH}(\mathcal{F}), R) \longrightarrow G(\text{Sh})(X|\mathcal{F}, R)$$

and the canonical graded morphism

$$\psi: \widetilde{\text{Sh}}(X|\mathcal{F}, R) \longrightarrow G(\text{Sh})(X|\mathcal{F}, R)$$

induce homomorphisms of corresponding homology modules φ_* and ψ_* , respectively. Let $E_*(\text{Sh}(X|\mathbf{LH}(\mathcal{F}), R))$ denote the graded R - R -submodule $\varphi_*^{-1}(\text{Im} \psi_*)$ of $H_*(\text{Sh}(X|\mathbf{LH}(\mathcal{F}), R))$. Homogeneous elements in $E_*(\text{Sh}(X|\mathbf{LH}(\mathcal{F}), R))$ and the cycles representing them are called *extendable classes* and *cycles* respectively. Focusing on the first homology and tracing along the diagram

$$\begin{array}{ccccccc} \rightarrow & \text{Sh}_2(X|\mathbf{LH}(\mathcal{F}), R)_p & \xrightarrow{D_{2p}} & \text{Sh}_1(X|\mathbf{LH}(\mathcal{F}), R)_p & \xrightarrow{D_{1p}} & \text{Sh}_0(X|\mathbf{LH}(\mathcal{F}), R)_p & \rightarrow \\ & \varphi_{2p} \downarrow & & \varphi_{1p} \downarrow & & \varphi_{0p} \downarrow & \\ \rightarrow & G(\text{Sh})_2(X|\mathcal{F}, R)_p & \xrightarrow{G(d_2)_p} & G(\text{Sh})_1(X|\mathcal{F}, R)_p & \xrightarrow{G(d_1)_p} & G(\text{Sh})_0(X|\mathcal{F}, R) & \rightarrow \\ & \psi_{2p} \uparrow & & \psi_{1p} \uparrow & & \psi_{0p} \uparrow & \\ \rightarrow & \widetilde{\text{Sh}}_2(X|\mathcal{F}, R)_p & \xrightarrow{\tilde{d}_{2p}} & \widetilde{\text{Sh}}_1(X|\mathcal{F}, R)_p & \xrightarrow{\tilde{d}_{1p}} & \widetilde{\text{Sh}}_0(X|\mathcal{F}, R)_p & \rightarrow \end{array}$$

a homological characterization of the general PBW property is obtained as a special case of ([Gol], Theorem 1).

3.1. Theorem With notation as before, the following statements are equivalent.

- (i) $\langle \mathbf{LH}(\mathcal{F}) \rangle = \langle \mathbf{LH}(I) \rangle$, that is, $G(A) \cong R/\langle \mathbf{LH}(\mathcal{F}) \rangle$.
- (ii) $E_1(\text{Sh}(X|\mathbf{LH}(\mathcal{F}), R)) = H_1(\text{Sh}(X|\mathbf{LH}(\mathcal{F}), R))$.
- (iii) The R - R -bimodule $H_1(\text{Sh}(X|\mathbf{LH}(\mathcal{F}), R))$ is generated by extendable classes.

4. Examples

The obvious application of previous sections 1–2 may be seen from Remark (ii) of section 1. In consideration of Koszulity, we finish this paper with two examples.

Let $R = \bigoplus_{p \in \mathbb{N}} R_p$ be a path algebra defined by a finite directed graph (or let R be a finitely generated free algebra) over a field K , where the positive gradation is defined by the lengths of paths. If I is generated by homogeneous elements of degree 2, then $A = R/I$ is called a *quadratic algebra*. One of the themes in the study of quadratic algebra has been the Koszulity (the well-known fact is that if $A = R/I$ is Koszul in the classical sense, then I is generated necessarily by homogeneous elements of degree 2). Applying noncommutative Gröbner basis theory to R , if, with respect to a fixed monomial ordering \prec on the standard K -basis \mathcal{B} of R , the reduced Gröbner basis of I (it always exists) consists of quadratic homogeneous elements, then A is Koszul (for instance, see [GH]). Combined with the N -type PBW property, the N -Koszulity in the sense of [Ber] is generalized to ungraded quotients ([BG2], Definition 3.9), that is, taking the grading filtration FR on R into account, for $P \subset F_N R$, $N \geq 2$, and $I = \langle P \rangle$, the algebra $A = R/I$ is said to be Koszul if the graded algebra $R/\langle \mathbf{LH}(P_N) \rangle$ is N -Koszul and if the N -type PBW property holds (see section 0 for the notation used here).

4.1. Example Let I be an ideal of R , and let $\mathcal{G} = \{g_j\}_{j \in J}$ be a Gröbner basis for I with respect to some graded monomial ordering \succeq_{gr} on \mathcal{B} . Consider the grading filtration FR on R and the induced filtration FA on the quotient algebra $A = R/I$. With notation maintained from previous sections, the following statements hold.

- (i) A has the general PBW property in the sense of Definition 1.8, that is, the associated graded algebra $G(A)$ is isomorphic to $R/\langle \mathbf{LH}(\mathcal{G}) \rangle$. Moreover, $\mathbf{LH}(\mathcal{G})$ is a Gröbner basis for $\langle \mathbf{LH}(I) \rangle$.
- (ii) If $\mathcal{G} \subset F_2 R$ and $\mathbf{LH}(\mathcal{G}) \neq 0$, then $G(A)$ is Koszul in the classical sense; If $\mathcal{G} \subset F_N R$ for $N \geq 2$ such that $\mathbf{LH}(\mathcal{G}) \neq 0$, then A is Koszul in the sense of [BG2] whenever $R/\langle \mathbf{LH}(\mathcal{G}) \rangle$ is N -Koszul in the sense of [Ber].
- (iii) The Rees algebra \tilde{A} of A is isomorphic to $R[t]/\langle \mathcal{G}^* \rangle$. Moreover, \mathcal{G}^* is a Gröbner basis for I^* .
- (iv) In the case that $\mathcal{G} \subset F_2 R$, \tilde{A} is Koszul in the classical sense.

4.2. Example Consider any quadric solvable polynomial algebra A studied in [Li] (in particular, Examples (i)–(vi) constructed in section 2 of Chapter III). Then A has the following properties.

- (a) A has the general PBW Property in the sense of Definition 1.8 (indeed they all have classical standard PBW bases).
- (b) With respect to its natural filtration FA (induced by the grading filtration of a free algebra), $G(A)$ is Koszul in the classical sense.
- (c) A is Koszul in the sense of [BG2].
- (d) The Rees algebra \tilde{A} of A with respect to FA is a classical Koszul algebra.

Final remark The result of Theorem 1.6 can be generalized to consider quotient algebras of a Γ -graded algebra R with the Γ -grading filtration $\mathcal{F}R$, where Γ is an ordered semigroup with a total ordering \prec . For instance, let $R = K\langle x_1, \dots, x_n \rangle$ be a finitely generated free K -algebra over a field K , and let \mathcal{B} be the standard K -basis of R consisting of words of length ≥ 0 . If \prec is a monomial ordering on \mathcal{B} , then, noticing now $R = \oplus_{u \in \mathcal{B}} Ku$ is \mathcal{B} -graded, we may consider the grading \mathcal{B} -filtration $\mathcal{F}R$ of R . Let I be a two-sided ideal of R and $A = R/I$. Then $\mathcal{F}R$ induces a \mathcal{B} -filtration $\mathcal{F}A$ for A that defines the associated \mathcal{B} -graded algebra $G^{\mathcal{F}}(A)$ of A . In a similar way, we can reach an analogue of Theorem 1.6, that is, $G^{\mathcal{F}}(A)$ is isomorphic to the *monomial algebra* $R/\langle \mathbf{LM}(I) \rangle$, where $\mathbf{LM}(I)$ is the set of all leading monomials of I . If furthermore I is generated by a Gröbner basis \mathcal{G} , then $G^{\mathcal{F}}(A) \cong R/\langle \mathbf{LM}(\mathcal{G}) \rangle$. A detailed discussion on this result and its applications will be given in a forthcoming paper.

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